TUG-OF-WAR WITH NOISE AND AN INVARIANCE OF p-HARMONIC FUNCTIONS UNDER BOUNDARY PERTURBATIONS

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ABSTRACT. In this paper, we provide new results about an invariance of p-harmonic functions under boundary perturbations by using tug-of-war with noise; a probabilistic interpretation of p-harmonic functions introduced by Peres-Sheffield in [PS08]. As a main result, when $E \subset \partial \Omega$ is countable and $f \in C(\partial \Omega)$, we provide a necessary and sufficient condition for E to guarantee that $H_g = H_f$ whenever g = f on $\partial \Omega \setminus E$. Here H_f and H_g denote the Perron solutions of f and g. It turns out that E should be of p-harmonic measure zero with respect to Ω . As a consequence, we analyze a structure of a countable set of p-harmonic measure zero. In particular, we give some results for the subadditivity of p-harmonic measures and an invariance result for p-harmonic measures. In addition, the results in this paper solve the problem regarding a perturbation point Björn [Bjö10] suggested for the case of unweighted \mathbb{R}^n .

1. Introduction

A function u on a domain Ω is called p-harmonic in Ω (for 1) if it is a weak solution to

$$\Delta_p u := \operatorname{div}(|Du|^{p-2}Du) = 0 \text{ in } \Omega,$$

(or as viscosity solutions—see either [JLM01] or Section 1 in [PS08]). That is, u is p-harmonic in Ω if and only if it belongs to the Sobolev space $W^{1,p}_{loc}(\Omega)$ (i.e., $\nabla u \in L^p_{loc}(\Omega)$) and

$$\int_{\Omega} |\nabla u|^{p-2} (\nabla u, \nabla \phi) dx = 0$$

for every $\phi \in C_0^{\infty}(\Omega)$. Δ_p is called the *p-Laplace operator* or *p-Laplacian*.

The Dirichlet problem for the p-Laplace equation involves finding a p-harmonic extension u to Ω of a boundary function f defined on $\partial\Omega$;

$$\Delta_p u = 0 \text{ in } \Omega \text{ and } u = f \text{ on } \partial \Omega.$$
 (1.1)

The existence and uniqueness of the solution for (1.1) is well-known in the Sobolev sense. (See [HKM06].) However, due to non-linearity of the p-Laplacian, there are many open problems. An intriguing problem is that of p-harmonic measure which is the solution of (1.1) when $f = \chi_E$ and $E \subset \partial\Omega$. More precisely, the p-harmonic measure of E with respect to Ω evaluated at $x \in \Omega$ is defined by

$$\omega_p(x; E, \Omega) = \overline{H}_{\chi_E}(x)$$

where \overline{H} denotes the upper Perron solution of (1.1). (See Section 2 for all the definitions and notations.) It is well known that when p=2 and Ω is regular, $\omega_p(x;\cdot,\Omega)$ defines a probability measure on $\partial\Omega$, but when $p\neq 2$, p-harmonic measure is not a measure. Very little is known about the measure theoretic properties of p-harmonic measure. Martio [Mar89] asked whether p-harmonic measure defines an outer measure on zero-level set of p-harmonic measure, i.e. whether p-harmonic measure is subadditive on the sets whose p-harmonic measure is zero. Llorente-Manfredi-Wu [LMW05] negatively answered to Martio's question; when Ω is the upper half plane, there exist sets $A, B \subset \partial\Omega$ such that $\omega_p(A,\Omega) = \omega_p(B,\Omega) = 0$, $A \cup B = \partial\Omega = \mathbb{R}$ and $|\mathbb{R} \setminus A| = |\mathbb{R} \setminus B| = 0$ where $|\cdot|$ stands for Lebesgue measure on \mathbb{R} . However, as far as the author is aware, the following problem concerning p-harmonic measure still remains unsolved.

Open Problem 1.1. When $E, F \subset \partial \Omega$ are both compact and $\omega_p(E, \Omega) = \omega_p(F, \Omega) = 0$, is it

$$\omega_p(E \cup F, \Omega) = 0$$
?

Further questions and discussions on p-harmonic measures can be found in [HKM06], [Bae97] and [BBS06].

Another interesting problem for (1.1) is a boundary perturbation problem; when f, g are two boundary functions on $\partial\Omega$ such that f=g except $E\subset\partial\Omega$, what condition for E implies $H_f=H_g$? (Here H_f and H_g denotes the Perron solutions of f and g.) When $\Omega\subset\mathbb{R}^n$ is bounded and $1< p\leq n$, an important result is obtained by Björn-Björn-Shanmugalingam [BBS03]; if $f\in C(\partial\Omega)$ and g=f on $\partial\Omega$ except a set of p-capacity zero, then $H_f=H_g$. Note that when $\Omega\subset\mathbb{R}^n$ and p>n, there exists no set of p-capacity zero. Therefore the methods in [BBS03] cannot be applied when p>n. There has been little work done when p>n. Even when n=2 and p>2, a seemingly simple question suggested by Baernstein [Bae97] has not been answered until the works of Björn [Bjö10] and Kim-Sheffield [KS09]; if $\Omega=B(0,1)\subset\mathbb{R}^2$, E is a finite union of open arcs on $\partial\Omega$, $f=\chi_E$ and $g=\chi_{\overline{E}}$, then $H_f=H_g$. A first result for a boundary perturbation problem when $n\geq 2$ and p>n is given by Björn [Bjö10], where he introduced the notion of a perturbation point which is a simple version of a boundary perturbation problem;

Definition 1.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. $x_0 \in \partial\Omega$ is called a perturbation point $(of \Omega)$; whenever $f \in C(\partial\Omega)$ and g is a bounded function on $\partial\Omega$ such that g = f on $\partial\Omega \setminus \{x_0\}$, we have

$$H_f = H_g$$
.

Note that not every regular boundary point is a perturbation point as the following example shows.

Example 1.3. Let $n and <math>\Omega = B(0,1) \setminus \{0\} \subset \mathbb{R}^n$. Let f = 0 and $g = \chi_{\{0\}}$ on $\partial \Omega$. Then we can verify that $H_f = 0$ and $H_g(x) = 1 - |x|^{\frac{p-n}{p-1}}$. Therefore, $H_f \neq H_g$ and 0 is not a perturbation point.

As one major result in [Bjö10], Björn showed that an exterior ray point is always a perturbation point and $H_f = H_g$ whenever $f \in C(\partial\Omega)$ and g = f on $\partial\Omega$ except countable exterior ray points. Most of the results in [Bjö10] can be extended by replacing an exterior ray point with any perturbation point. By observing that 0 in Example 1.3 is an isolated boundary point, Björn proposed the following problem in [Bjö10];

Björn's problem) Is it true that any regular point which is not isolated among the regular boundary points is a perturbation point?

In this paper, we give several invariance results for p-harmonic functions including an affirmative answer to Björn's problem by using tug-of-war with noise; a probabilistic interpretation of p-harmonic functions introduced by Peres-Sheffield in [PS08]. The main result is Theorem 5.3, which reveals a link between p-harmonic measure and a boundary perturbation problem as well as analyzes the structure of a countable set of p-harmonic measure zero and gives a necessary and sufficient condition for a boundary perturbation problem when $f \in C(\partial\Omega)$ and $E \subset \partial\Omega$ is countable. An interesting fact is that when $E \subset \partial\Omega$ is a countable set, a boundary perturbation problem and $\omega_p(E,\Omega) = 0$ are local properties. As other important consequences, Theorem 5.4 and Theorem 5.5 show that p-harmonic measure is subadditive on $\{E \subset \partial\Omega : E \text{ is a countable set of } p$ -harmonic measure zero does not affect the p-harmonic measure of any closed set on $\partial\Omega$. Theorem 3.9 and Theorem 4.1 will play a vital role to obtain most of results. In particular, Theorem 4.1 answers Björn's problem affirmatively and shows the locality of a perturbation point. All the results are new when p > n.

The outline of the paper is as follows. In Section 2, we give some preliminary results for p-harmonic functions and Perron solutions. In Section 3, we give a brief explanation of tug-of-war with noise and characterize a perturbation point in terms of tug-of-war with noise. In Section 4, we give a necessary and sufficient condition for a perturbation point, thereby answering Bjorn's question affirmatively. As applications, in Section 5, we provide several results for p-harmonic measures as well as a boundary perturbation problem with a countable set. Finally, in Section 6, we give some open problems concerning a boundary perturbation problem and p-harmonic measures.

2. Definitions and preliminary results

The main reference for the results and notation in this section is [HKM06].

Definition 2.1. A domain $\Omega \subset \mathbb{R}^n$ is an open connected subset. When there exists $B(0,R) \subset \mathbb{R}^n$ such that $\Omega \subset B(0,R)$, we say that Ω is a bounded domain.

First, we state some properties of p-harmonic functions which will be used later in this paper.

Theorem 2.2. (Strong maximum principle) A nonconstant p-harmonic function in a domain Ω cannot attain its supremum or infimum.

Theorem 2.3. (Harnack's convergence theorem) Suppose that $u_i, i = 1, 2, ..., is$ an increasing sequence of p-harmonic functions in Ω . Then the function $u = \lim_{i \to \infty} u_i$ is either p-harmonic in Ω or identically $+\infty$.

Definition 2.4. A function $u: \Omega \to (-\infty, \infty]$ is called *p-superharmonic* in Ω if (i) u is lower semicontinuous in Ω , (ii) $u \neq \infty$ in Ω , and (iii) for each domain $D \subset\subset \Omega$, the following comparison principle holds: if $h \in C(\overline{D})$ is *p*-harmonic in D and $u \geq h$ on ∂D , then $u \geq h$ in D. We say that u is *p-subharmonic* in Ω if -u is *p*-superharmonic in Ω .

The following comparison principle will be used many times throughout this paper.

Theorem 2.5. (Comparison Principle) Suppose that u is p-superharmonic and v is p-subharmonic in Ω . If

$$\limsup_{y\to x} v(y) \le \liminf_{y\to x} u(y)$$

for all $x \in \partial \Omega$, and also for $x = \infty$ if Ω is unbounded, (excluding the situation $\infty \leq \infty$ and $-\infty \leq -\infty$), then $v \leq u$ in Ω .

Definition 2.6. Let $f: \partial\Omega \to [-\infty, \infty]$. The upper class \mathcal{U}_f consists of all the functions u such that (i) u is p-superharmonic in Ω , (ii) u is bounded below, and (iii) $\lim \inf_{x\to y} u(x) \geq f(x)$ for all $y\in\partial\Omega$. The lower class \mathcal{L}_f is defined as $v\in\mathcal{L}_f$ if and only if $-v\in\mathcal{U}_{-f}$.

Definition 2.7. The upper Perron solution, \overline{H}_f and lower Perron solution, \underline{H}_f are defined by

$$\overline{H}_f(x) = \inf\{u(x) : u \in \mathcal{U}\} \text{ and } \underline{H}_f(x) = \sup\{v(x) : v \in \mathcal{L}\}.$$

Note that the comparison principle shows that $\underline{H}_f \leq \overline{H}_f$. We list some basic properties of the Perron solutions.

Proposition 2.8.

- i) \underline{H}_f and \overline{H}_f are p-harmonic in Ω unless they are not identically $\pm \infty$.
- ii) Let $f_j: \partial\Omega \to [-\infty, \infty)$ be a decreasing sequence of upper semicontinuous functions and $f = \lim_{f \to \infty} f_j$. Then $\overline{H}_f = \lim_{f \to \infty} \overline{H}_{f_i}$.

For the boundary continuity of the Perron solutions, we introduce the notion of regularity.

Definition 2.9. $x_0 \in \partial \Omega$ is called a *regular* point of Ω , if

$$\lim_{x \to x_0} \overline{H}_f(x) = f(x_0)$$

for each continuous function $f: \partial\Omega \to \mathbb{R}$. A point is *irregular* if it is not regular. If all boundary points of Ω are regular, then Ω is called *regular*.

A necessary and sufficient condition for regularity is well-known. (See Chapter 6 in [HKM06].) In particular, any Lipschitz domain is regular and when p > n, any domain is regular. It is natural to ask which one of the two Perron solutions \overline{H}_f and \underline{H}_f is the "correct" solution to the Dirichlet problem. We introduce the notion of resolutivity.

Definition 2.10. We say that f is resolutive if \underline{H}_f and \overline{H}_f agree. When f is resolutive, we denote the Perron solution by $H_f := \underline{H}_f = \overline{H}_f$ and call it the p-harmonic extension of f to Ω .

When p=2, it is known that all measurable functions are resolutive. It is an open question whether all measurable functions are resolutive for general p. However, the following result is known for resolutivity. For more details see Chapter 9 in [HKM06]

Theorem 2.11. Let Ω be regular. If f is bounded and lower(or upper) semicontinuous on $\partial\Omega$, then f is resolutive in Ω .

Remark: Theorem 2.11 and Theorem 2.19 shows that any bounded function which is continuous except a single point is resolutive. Therefore, H_g is well-defined in Definition 1.2.

Now let us define p-harmonic measure by the upper Perron solution.

Definition 2.12. The function $\omega_p(x, E, \Omega) = \overline{H}_{\chi_E}(x) = \inf \mathcal{U}_E$ is called the *p*-harmonic measure of $E \subset \partial \Omega$ at $x \in \Omega$ with respect to Ω . If $\omega_p(E, \Omega) = 0$, we say that E is of *p*-harmonic measure zero.

Proposition 2.13.

- i) $0 \le \omega_p(x, E, \Omega) \le 1$. Furthermore, if $\omega_p(x, E, \Omega) = 0$ at some $x \in \Omega$, then $\omega_p(x, E, \Omega) \equiv 0$ in Ω .
- ii) If $E_1 \subset E_2 \subset \partial\Omega$, then $\omega_p(x, E_1, \Omega) \leq \omega_p(x, E_2, \Omega)$.
- iii) If $E \subset \partial \Omega_1 \cap \partial \Omega_2$ and if $\Omega_1 \subset \Omega_2$, then $\omega_p(x, E, \Omega_1) \leq \omega_p(x, E, \Omega_2)$ in Ω_1 .

To Open problem 1.1, there is a partial answer. (See Theorem 11.17 in [HKM06].)

Theorem 2.14. Let $1 and let <math>\Omega$ be regular. If $E, F \subset \partial \Omega$ are closed sets of p-harmonic measure zero and $E \cap F = \phi$, then $\omega_p(E \cup F, \Omega) = 0$.

Next we introduce a notion of p-capacity.

Definition 2.15. The *p-capacity* of E is defined by

$$C_p(E) := \inf \int_{\mathbb{R}^n} (|u|^p + |\nabla u|^p)$$

where the infimum is taken over all $u \in W^{1,p}(\mathbb{R}^n)$ such that u = 1 in a neighborhood of E. If $C_p(E) = 0$, we say that E is a set of p-capacity zero.

Here are some basic properties of p-capacity and see Chapter 2 in [HKM06] for more properties.

Proposition 2.16.

- i) A point of \mathbb{R}^n is of *p*-capacity zero if and only if 1 . In particular, when <math>p > n, there exists no nonempty set of *p*-capacity zero.
- ii) $C_p(\sum_i E_i) \leq \sum_i C_p(E_i)$. In particular, when 1 , every countable set is of p-capacity zero.

A set of p-capacity zero can be described in terms of p-harmonic measure.

Definition 2.17. We say that $E \subset \mathbb{R}^n$ is of absolute p-harmonic measure zero if $\omega_p(E \cap \partial\Omega,\Omega) = 0$ for all bounded domains $\Omega \subset \mathbb{R}^n$.

We state Theorem 11.15 in [HKM06].

Theorem 2.18. E is of absolute p-harmonic measure zero if and only if E is of p-capacity zero.

When $1 and <math>E \subset \partial\Omega$ is of p-capacity zero, Björn-Björn-Shanmugalingam showed the following result for a boundary perturbation problem.

Theorem 2.19. (Björn-Björn-Shanmugalingam [BBS03]) Assume that $f \in C(\partial\Omega)$ and g = f on $\partial\Omega$ except a set of p-capacity zero. Then g is resolutive and

$$H_g = H_f$$
.

Corollary 2.20. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded domain. Every point on $\partial\Omega$ is a perturbation point.

Definition 2.21. We say that $x_0 \in \partial \Omega$ is an *exterior ray point* if there is a line segment, \mathcal{L} such that $x_0 \in \mathcal{L}$ and $\mathcal{L} \subset \mathbb{R}^n \setminus \Omega$.

For instance, if $\Omega = B(0,1) \setminus \{0 < x < 1\} \subset \mathbb{R}^n$, 0 is an exterior ray point.

Theorem 2.22. (Björn [Bjö10]) Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded domain. An exterior ray point is a perturbation point.

Theorem 2.23. (Björn [Bjö10]) Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $E \subset \partial \Omega$ be a countable set whose elements are perturbation points of Ω . If $f \in C(\partial \Omega)$ and g = f on $\partial \Omega \setminus E$, then g is resolutive and

$$H_g = H_f. (2.1)$$

In particular, when E consists of exterior ray points, (2.1) holds.

Note that a major part in Theorem 2.23 is when p > n. When 1 , Theorem 2.23 is just a consequence of Theorem 2.19 because the <math>p-capacity of a countable set is always zero. Also note that we neither require g to be bounded nor to be continuous on $\{x \in \partial\Omega : g(x) = f(x)\}$.

3. Tug-of-war with noise and game-perturbation points

When p=2, it is discovered by Kakutani [Kak44] that the Dirichlet problem can be solved in a probabilistic way; $u(x) = \mathbb{E}_x(f(B_\tau))$ where \mathbb{E}_x stands for the expected value when a Brownian motion B starts at x and runs until hitting time τ of $\partial\Omega$. However, when $p \neq 2$, a probabilistic interpretation of p-harmonic functions has remained unknown until recently Peres-Sheffield's works. (See also [MPR09].) Their works were initiated to figure out the behaviors of two-player random turn games like a random turn hex [PSSW07]. After some further research, they found that the value of a two-player random turn game is related to the ∞ -Laplace equation, $\Delta_\infty u := |\nabla u|^{-2} \Sigma_{i,j} u_i u_{i,j} u_j = 0$, and named the game tug-of-war [PSSW09]. By noticing $\Delta_p u = |\nabla u|^{p-2} \{\Delta u + (p-2)\Delta_\infty u\}$, they finally showed that a variant of tug-of-war, called tug-of-war with noise, gives a probabilistic solution to the Dirichlet problem (1.1).

In this section, we give a quick summary of tug-of-war with noise and apply it to characterize a perturbation point in a probabilistic way.

Tug-of-war with noise Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $\alpha = 1 + \sqrt{(n-1)/(p-1)}$ and let $f: \partial \Omega \to \mathbb{R}$ be the terminal payoff function. The game is played as follows: At the kth step, a fair coin is tossed, and the winning player is allowed to make a move v with $|v| \leq \epsilon$. If $\operatorname{dist}(x_{k-1}, \partial \Omega) > \alpha \epsilon$, then the moving player chooses $v_k \in \mathbb{R}^n$ with $|v_k| \leq \epsilon$ and sets $x_k = x_{k-1} + v_k + z_k$ where z_k is a random "noise" vector whose law is the uniform distribution on the sphere of radius $|v_k| \sqrt{(n-1)/(p-1)}$ in the hyperplane orthogonal to v_k . (Here we chose a simple noise vector. See [PS08] for more details of a noise vector.) If $\operatorname{dist}(x_{k-1}, \partial \Omega) \leq \alpha \epsilon$, then the moving player chooses an $x_k \in \partial \Omega$ with $|x_k - x_{k-1}| \leq \alpha \epsilon$ and the game ends, with player I receiving a payoff of $f(x_k)$ from player II. Both players receive a payoff of zero if the game never terminates.

Definition 3.1. A strategy for players is a way of choosing the player's next move as a function of all previously played moves and all coin tosses. More precisely it is a sequence of Borel-measurable maps from $\Omega \times (\overline{B(0,\varepsilon)} \times \overline{\Omega})^k$ to $\overline{B(0,\varepsilon)}$, giving the move a player would make at the kth step of the game as a function of the game history.

Note that a pair of strategies $\sigma = (S_I, S_{II})$ (where S_I is a strategy for player I and S_{II} is a strategy for player II) and a starting point x determine a unique probability measure \mathbb{P}_x on the space of game position sequences. Let us denote the corresponding expectation by \mathbb{E}_x .

Definition 3.2. The value of the game for player I at x is defined by $u_1^{\epsilon}(x) = \sup_{S_I} \inf_{S_{II}} V_x(S_I, S_{II})$ and the value of the game for player II at x is defined by $u_2^{\epsilon}(x) = \inf_{S_{II}} \sup_{S_I} V_x(S_I, S_{II})$ where $V_x(S_I, S_{II}) = \mathbb{E}_x \left[f(x_\tau) \chi_{\{\tau < \infty\}} \right]$ is the expected payoff and τ is the exit time of Ω .

By definitions, we always have $u_1^{\epsilon}(x) \leq u_2^{\epsilon}(x)$.

Definition 3.3. $x_0 \in \partial \Omega$ is called a game-regular point of Ω if for every $\delta > 0$ and $\eta > 0$ there exists a δ_0 and ϵ_0 such that for every $x \in \Omega \cap B(x_0, \delta_0)$ and $\epsilon < \epsilon_0$, player I has a strategy that guarantees that an ϵ -step game started at x will terminate at a point on $\partial \Omega \cap B(x_0, \delta)$ with probability at least $1 - \eta$. Ω is game-regular if every $x \in \partial \Omega$ is game-regular.

The main results in [PS08] are the followings.

Theorem 3.4. (Peres-Sheffield [PS08]) Let $1 and let <math>\Omega$ be a bounded domain in \mathbb{R}^n .

- i) If p > n, then Ω is game-regular.
- ii) If Ω satisfies an exterior cone condition at every point $x \in \partial \Omega$, then Ω is gameregular.
- iii) If n=2 and Ω is simply connected, then Ω is game-regular.

Theorem 3.5. (Peres-Sheffield [PS08]) Let $\Omega \subset \mathbb{R}^n$ be a bounded game-regular domain and f be a continuous function on $\partial\Omega$. Then as $\epsilon \to 0$, the game values u_1^{ϵ} and u_2^{ϵ} converge uniformly to the unique p-harmonic function u that extends continuously to f on $\partial\Omega$.

Corollary 3.6. Let $\Omega \subset \mathbb{R}^n$ be a bounded game-regular domain. Then Ω is also regular.

Let us think about a probabilistic meaning of a perturbation point in terms of tugof-war with noise. If a boundary point is a perturbation point, it means that the payoff value at that point does not affect the game value. Therefore, it is naturally guessed that a perturbation point should be avoidable with high probability by one player whatever the other player does. This insight makes us define the following notion.

Definition 3.7. $x_0 \in \partial \Omega$ is called a game-perturbation point of Ω if for every $\delta > 0$ and $\eta > 0$ there exist a δ_0 and ϵ_0 such that for every $x \in \Omega \cap B(x_0, \delta_0)$ and $\epsilon < \epsilon_0$, player I has a strategy that guarantees that an ϵ -step game started at x will terminate at a point on $\partial \Omega \cap B(x_0, \delta) \setminus B(x_0, \delta_x)$ with probability at least $1 - \eta$ and some δ_x which is a constant depending on x with $0 < \delta_x < \delta$.

The following lemma will be very useful to a game-theoretic proof of the results in this paper.

Lemma 3.8. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded game-regular domain. Suppose that $f: \partial\Omega \to [0,1]$ is continuous. Let $x \in \Omega$ and $\eta > 0$. Then there exists a $\varepsilon_0 > 0$ such that for every $\epsilon < \epsilon_0$, player I has a strategy that guarantees that an ϵ -step game started at x will terminate at a point on $\{y \in \partial\Omega : f(y) > 0\}$ with probability at least $H_f(x) - \eta$.

Proof. Theorem 3.5 shows that there exists a $\varepsilon_0 > 0$ such that for every $\varepsilon \leq \varepsilon_0$, $u_1^{\epsilon}(x) \geq H_f(x) - \eta/2$. Since $u_1^{\epsilon}(x) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}_x[f(x_{\tau})\chi_{\{\tau < \infty\}}]$, player I has a strategy which

guarantees that $\inf_{S_{II}} \mathbb{E}_x[f(x_\tau)\chi_{\{\tau<\infty\}}] \geq u_1^{\epsilon}(x) - \eta/2$. Note that

$$\mathbb{E}_{x}[f(x_{\tau})\chi_{\{\tau<\infty\}}] = \mathbb{E}_{x}[f(x_{\tau})\chi_{\{\tau<\infty\}}, x_{\tau} \in \{y \in \partial\Omega : f(y) > 0\}]$$

$$\leq \mathbb{P}_{x}(x_{\tau} \in \{y \in \partial\Omega : f(y) > 0\}).$$

Therefore, for any player II's strategy,

$$\mathbb{P}_x(x_\tau \in \{y \in \partial\Omega : f(y) > 0\}) \ge u_1^{\epsilon}(x) - \eta/2 \ge H_f(x) - \eta.$$

Now we are ready to provide a probabilistic characterization of a perturbation point by using tug-of-war with noise.

Theorem 3.9. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded game-regular domain. For $x_0 \in \partial \Omega$, the following conditions are equivalent.

- i) x_0 is a perturbation point.
- ii) x_0 is a game-perturbation point.

Proof. First note that Ω is also regular by Corollary 3.6.

 $i) \Rightarrow ii)$ Fix $\delta > 0$ and $\eta > 0$. Define $f: \partial\Omega \to [0,1]$ as a function such that f=1 on $\partial\Omega \cap B(x_0,\delta/2), \ f=0$ on $\partial\Omega \setminus B(x_0,\delta)$, otherwise f is continuous. By the regularity of x_0 , there exists a $\delta_0 > 0$ such that whenever $x \in \Omega \cap B(x_0,\delta_0), \ H_f(x) \geq 1-\eta/3$. Let $\tilde{x} \in \Omega \cap B(x_0,\delta_0)$. Let us construct an increasing sequence $\{g_n\}$ of lower-semicontinuous functions on $\partial\Omega$ by letting $g_n=0$ on $\partial\Omega \cap \overline{B(x_0,\delta_0/n)}$, otherwise $g_n=f$. Note that g_n is resolutive by Theorem 2.11. Proposition 2.8 shows that $\lim H_{g_n}(\tilde{x}) = \lim H_g(\tilde{x})$ where g is a function on $\partial\Omega$ such that g=f on $\partial\Omega \setminus \{x_0\}$ and $g(x_0)=0$. Therefore, there exists a N such that $H_{g_N}(\tilde{x}) \geq H_g(\tilde{x}) - \eta/3$. Let $\delta_x = \delta_0/2N$. Let $h:\partial\Omega \to [0,1]$ be a continuous function such that $h \geq g_N$ on $\partial\Omega$, $h = g_N$ on $\partial\Omega \setminus \overline{B(x_0,\delta_0/N)}$ and h = 0 on $\partial\Omega \cap B(x_0,\delta_0/2N)$. Since $H_h(\tilde{x}) \geq H_{g_N}(\tilde{x})$, it follows that $H_h(\tilde{x}) \geq H_g(\tilde{x}) - \eta/3$. Note that $H_g = H_f$ because x_0 is a perturbation point. Since $H_f(\tilde{x}) \geq 1 - \eta/3$, it follows that $H_h(\tilde{x}) \geq 1 - 2\eta/3$. Lemma 3.8 shows that player I has a strategy that guarantees that for some ε_0 , an ϵ -step game started at \tilde{x} with $\varepsilon \leq \varepsilon_0$ will terminate at a point on $\{x \in \partial\Omega : h(x) > 0\}$ with probability at least $H_h(\tilde{x}) - \eta/3 \geq 1 - \eta$. Since $\{x \in \partial\Omega : h(x) > 0\} \subset \partial\Omega \cap B(x_0,\delta) \setminus B(x_0,\delta_0/2N)$, the proof is complete.

 $ii) \Rightarrow i)$ Let $f \in C(\partial\Omega)$ and g be a bounded function on $\partial\Omega$ such that g = f on $\partial\Omega \setminus \{x_0\}$. To prove that $H_f = H_g$, it is enough to show that $\lim_{x \in \Omega \to x_0} H_f(x) = \lim_{x \in \Omega \to x_0} H_g(x)$ by the comparison principle and the regularity of Ω . Fix $\eta > 0$. Since f is continuous at x_0 , there exists $\delta > 0$ such that for all $y \in \partial\Omega \cap B(x_0, \delta)$, $|f(y) - f(x_0)| \leq \eta$. Let $x \in \partial\Omega \cap B(x_0, \delta_0)$ and let $M = \sup_{\partial\Omega} (|f| + |g|)$. Let $g_M : \partial\Omega \to \mathbb{R}$ be a continuous function such that $g_M = f$ on $\partial\Omega \setminus B(x_0, \delta_x)$, $g_M \leq f$ on $B(x_0, \delta_x)$ and $g_M(x_0) = -M$ where δ_x is given from the assumption that x_0 is a game-perturbation point. Denote by $u_{g_M}^{1,\epsilon}(x)$ the game value for player I at x with the payoff function g_M . Since x_0 is a game-perturbation point, player I has a strategy which guarantees that for some $\varepsilon_0 > 0$, whenever $\varepsilon \leq \varepsilon_0$, $u_{g_M}^{1,\epsilon}(x) \geq f(x_0) - M\eta$. Letting $\varepsilon \to 0$,

Theorem 3.5 shows that $H_{g_M}(x) \geq f(x_0) - M\eta$. Since $H_g(x) \geq H_{g_M}(x)$ and x is an arbitrary point on $\partial\Omega \cap B(x_0, \delta_0)$, $\liminf_{x \in \Omega \to x_0} H_g(x) \geq f(x_0) - M\eta$. Letting $\eta \to 0$ shows that $\liminf_{x \in \Omega \to x_0} H_g(x) \geq f(x_0)$. Since x_0 is a regular boundary point of Ω , $\lim_{x \in \Omega \to x_0} H_f(x) = f(x_0)$. Therefore, $\liminf_{x \in \Omega \to x_0} H_g(x) \geq \lim_{x \in \Omega \to x_0} H_f(x)$. Similarly, player II adopting the strategy in i) shows that $\limsup_{x \in \Omega \to x_0} H_g(x) \leq \lim_{x \in \Omega \to x_0} H_f(x)$. Therefore, $\lim_{x \in \Omega \to x_0} H_f(x) = \lim_{x \in \Omega \to x_0} H_g(x)$ and the proof is complete.

Corollary 3.10. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded game-regular domain. Then every $x_0 \in \partial \Omega$ is a game-perturbation point.

Proof. The result follows from Theorem 3.9 and Corollary 2.20.

4. Characterization of Perturbation Points

In this section, we provide a necessary and sufficient condition for a perturbation point. As Corollary 2.20 shows, our main concern for a perturbation point is the case of p > n. Together with Theorem 3.9, the following theorem will be a cornerstone.

Theorem 4.1. Let p > n and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $x_0 \in \partial \Omega$. Then the following conditions are equivalent.

- i) x_0 is a perturbation point.
- ii) x_0 is a game-perturbation point.
- iii) There exists $\{x_k\}$ such that for all $k \in \mathbb{N}$, $x_k \neq x_0$, $x_k \in \mathbb{R}^n \setminus \Omega$ and $\lim_k x_k = x_0$.
- vi) $\omega_p(\{x_0\}, \Omega) = 0.$

Proof. We prove our statement by showing $vi) \Rightarrow iii$, $iii) \Rightarrow ii$, $iii) \Rightarrow i$, and $iii) \Rightarrow vi$.

- $vi) \Rightarrow iii$) Suppose that iii) is not true. Then there exists $B(x_0, \delta) \subset \Omega$ with some $\delta > 0$. Note that if p > n, any bounded domain in \mathbb{R}^n is game-regular by Theorem 3.4. Therefore, Theorem 3.5 implies that $\lim_{x \in \Omega \to x_0} \omega_p(x; \{x_0\}, \Omega) = 1$, which contradicts to $\omega_p(\{x_0\}, \Omega) = 0$.
- $iii) \Rightarrow ii)$ The key idea is using an iteration to find a game-perturbation strategy for player I. Without loss of generality, we can assume that $x_0 = 0$. Therefore, there exists $\{x_k\}$ such that for all $k \in \mathbb{N}$, $x_k \neq 0$, $x_k \in \mathbb{R}^n \setminus \Omega$ and $\lim_k x_k = 0$. Inductively we construct a subsequence of $\{x_k\}$, $\{y_k\}$ such that $|y_k|$ is decreasing to 0 and

$$\frac{|y_k|}{|y_{k+1}|} \le \frac{|y_{k+1}|}{|y_{k+2}|} \text{ for all } k \in \mathbb{N}.$$

Suppose that we have $\{y_i: 1 \leq i \leq k+1\}$. Then we choose y_{k+2} among $\{x_k\}$ as $|y_{k+2}| \leq \frac{|y_{k+1}|^2}{|y_k|}$. This can be done inductively because $\{x_k\}$ is converging to 0 and $x_k \neq 0$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let

$$\Omega_k = \{x \in \mathbb{R}^n : |y_{k+2}| < |x| < |y_k|\} \setminus \{y_{k+1}\}$$

and define a function $f_k: \Omega_k \to [0,1]$ as $f_k(x) = \omega_p(x; \{y_{k+1}\}, \Omega_k)$. Let $\theta_k = \inf_x \{f_k(x) : x \in \mathbb{R}^n, |x| = |y_{k+1}|\}$.

We show that as $k \to \infty$, θ_k is increasing, thereby $\theta_k \ge c > 0$ for all $k \in \mathbb{N}$ with some constant c. For this, note that

$$\begin{array}{ll} \theta_{k+1} & = & \inf_{x} \{f_{k+1}(x) : x \in \mathbb{R}^{n}, |x| = |y_{k+2}|\} \\ \\ & = & \inf_{x} \left\{ f_{k+1} \left(\frac{|y_{k+2}|}{|y_{k+1}|} x \right) : x \in \mathbb{R}^{n}, |x| = |y_{k+1}| \right\} \\ \\ & = & \inf_{x} \left\{ \omega_{p} \left(\frac{|y_{k+2}|}{|y_{k+1}|} x; \{y_{k+2}\}, \Omega_{k+1} \right) : x \in \mathbb{R}^{n}, |x| = |y_{k+1}| \right\} \\ \\ & = & \inf_{x} \left\{ \omega_{p} \left(x; \left\{ \frac{|y_{k+1}|}{|y_{k+2}|} y_{k+2} \right\}, \Omega'_{k} \right) : x \in \mathbb{R}^{n}, |x| = |y_{k+1}| \right\} \end{array}$$

where

$$\Omega'_{k} = \left\{ x \in \mathbb{R}^{n} : \frac{|y_{k+3}||y_{k+1}|}{|y_{k+2}|} < |x| < \frac{|y_{k+1}|^{2}}{|y_{k+2}|} \right\} \setminus \left\{ \frac{|y_{k+1}|}{|y_{k+2}|} y_{k+2} \right\}.$$

Here the last equality is obtained by the radial invariance of p-harmonic functions. In addition, a rotational invariance of p-harmonic functions shows that

$$\inf_{x} \left\{ \omega_{p} \left(x; \left\{ \frac{|y_{k+1}|}{|y_{k+2}|} y_{k+2} \right\}, \Omega_{k}' \right) : |x| = |y_{k+1}| \right\} = \inf_{x} \left\{ \omega_{p} \left(x; \left\{ y_{k+1} \right\}, \tilde{\Omega}_{k} \right) : |x| = |y_{k+1}| \right\}$$

where

$$\tilde{\Omega}_n = \left\{ x \in \mathbb{R}^n : \frac{|y_{k+3}||y_{k+1}|}{|y_{k+2}|} < |x| < \frac{|y_{k+1}|^2}{|y_{k+2}|} \right\} \setminus \{y_{k+1}\}.$$

Therefore, it follows that

$$\theta_{k+1} = \inf_{x} \left\{ \omega_p \left(x; \{ y_{k+1} \}, \tilde{\Omega}_k \right) : x \in \mathbb{R}^n, |x| = |y_{k+1}| \right\}.$$
 (4.1)

Since

$$\frac{|y_{k+3}||y_{k+1}|}{|y_{k+2}|} \le |y_{k+2}| \text{ and } \frac{|y_{k+1}|^2}{|y_{k+2}|} \ge |y_k|,$$

we have that $\Omega_k \subset \tilde{\Omega}_k$. Since $\{y_{k+1}\}\subset \partial \Omega_k \cap \partial \tilde{\Omega}_k$, Proposition 2.13 shows that

$$\omega_p\left(x;\{y_{k+1}\},\tilde{\Omega}_k\right) \ge \omega_p\left(x;\{y_{k+1}\},\Omega_k\right) = f_k(x).$$

It follows from (4.1) that $\theta_{k+1} \geq \theta_k$ for all $k \in \mathbb{N}$. Moreover, the minimum principle and the regularity of y_{k+1} (recall that if p > n, any domain in \mathbb{R}^n is regular) shows that $\theta_1 > 0$. Therefore, $\theta_k \geq \theta_1 > 0$ for all $k \in \mathbb{N}$.

Now we are ready to give a "game-perturbation strategy" for player I. First note that when p > n, every bounded domain in \mathbb{R}^n is game-regular by Theorem 3.4. Fix $\eta > 0$ and $\delta > 0$. We can find $i, j \in \mathbb{N}$ such that $(1 - \theta_1/2)^i < \eta$ and $|y_j| < \delta$. Let $\delta_0 = |y_{i+j}|$. Let $x_0 \in \Omega \cap B(0, \delta_0)$. Since $|y_k|$ is decreasing to 0, we can find some $N \in \mathbb{N}$ such that $x_0 \in B(0, |y_{i+j+N-1}|) \setminus B(0, |y_{i+j+N}|)$. The strategy for player I is the following; Let x_0 be an initial point and let $c = \omega_p(x_0; \{y_{i+j+N}\}, \Omega_{i+j+N-1})$. By the minimum principle, c > 0. Since $\Omega_{i+j+N-1}$ is game-regular and $\omega_p(x; \{y_{i+j+N}\}, \Omega_{i+j+N-1}) \in C(\overline{\Omega_{i+j+N-1}})$ is p-harmonic, Lemma 3.8 shows that player I has a strategy to guarantee that a sufficiently

small ϵ -step game position will arrive at y_{i+j+N} before hitting $\partial\Omega_{i+j+N-1}\setminus\{y_{i+j+N}\}$ with probability at least c/2. Note that since $y_{i+j+N}\in\mathbb{R}^n\setminus\Omega$, the game will terminate no later than the game position reaches y_{i+j+N} . Assume that the game position enters $B(0,|y_{i+j+N-1}|)$ before reaching y_{i+j+N} . Then, again by Lemma 3.8 with $f=\chi_{\{y_{i+j+N-1}\}}$ on $\Omega_{i+j+N-2}$, player I can arrange to reach $y_{i+j+N-1}$ before hitting $\partial\Omega_{i+j+N-2}\setminus\{y_{i+j+N-1}\}$ with probability at least $\theta_1/2>0$. Now we iterate this argument. Whenever the game position enters $B(0,|y_{k+1}|)$ with some $k\in\mathbb{N}$ before the game terminates, player I adopts a strategy given by Lemma 3.8 with $f=\chi_{\{y_{k+1}\}}$ on Ω_k . Therefore, iterating the above argument i times shows that player I has a strategy that guarantees that a sufficiently small ϵ -step game started at $x_0\in\Omega$ with $|x_0|<\delta_0$ will terminate at a point on $\{y_k: j\leq k\leq 2i+j+N\}$ with probability at least $1-(1-c/2)(1-\theta_1/2)^i>1-\eta$. Since $\{y_k: j\leq k\leq 2i+j+N\}$ with probability at least $1-(1-c/2)(1-\theta_1/2)^i>1-\eta$.

- $(ii) \Rightarrow i$) This is a part of the results in Theorem 3.9.
- $i) \Rightarrow iv$) This is the general property of a perturbation point. Let f = 0 and $g = \chi_{\{0\}}$. Then the result follows.

As an immediate result, we answer Björn's problem affirmatively.

Corollary 4.2. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that $x_0 \in \partial \Omega$ is not an isolated boundary point. Then x_0 is a perturbation point. In particular, $\omega_p(\{x_0\}, \Omega) = 0$.

Theorem 4.1 gives a necessary and sufficient condition for a perturbation point in terms of p-harmonic measure.

Theorem 4.3. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded domain. $x_0 \in \partial \Omega$ is a perturbation point if and only if $\omega_p(\{x_0\}, \Omega) = 0$.

Proof. When p > n, the result follows from Theorem 4.3. Assume that $1 . As Corollary 2.20 shows, every boundary point is a perturbation point. Therefore, we only need to show that <math>\omega_p(\{x_0\}, \Omega) = 0$. However, when 1 , every single point is of <math>p-capacity zero and $\omega_p(\{x_0\}, \Omega) = 0$ follows from Theorem 2.18.

In addition, when Ω is game-regular, we have the following.

Theorem 4.4. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded game-regular domain. For $x_0 \in \partial \Omega$, the following conditions are equivalent.

- i) $x_0 \in \partial \Omega$ is a perturbation point.
- ii) $x_0 \in \partial \Omega$ is a game-perturbation point.
- iii) $\omega_{p}(\{x_{0}\},\Omega)=0.$

Proof. The result follows from Theorem 3.9 and Theorem 4.3.

As other important consequence of Theorem 4.1, we show the locality of a perturbation point, which is not obvious from the definition of a perturbation point.

Theorem 4.5. Let $1 and let <math>\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be bounded domains. Let $x_0 \in \partial \Omega_1 \cap \partial \Omega_2$. Suppose that there exists an open neighborhood U of x_0 such that $U \cap \Omega_1 = U \cap \Omega_2$. Then x_0 is a perturbation point of Ω_1 if and only if x_0 is a perturbation point of Ω_2 .

Proof. By Corollary 2.20, the case of p > n is of our only concern. In that case, the result follows from ii) in Theorem 4.1.

5. Main results for perturbation sets and p-harmonic measures

In this section, we give a necessary and sufficient condition for a boundary perturbation problem when $f \in C(\partial\Omega)$ and E is countable. As we will see, it also characterize a structure of a countable set of p-harmonic measure zero. Theorem 5.3 is crucial. Before giving the result, we introduce two notions. First, we generalize the notion of a perturbation point to a set.

Definition 5.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. $E \subset \partial \Omega$ is called a perturbation set $(of \Omega)$; whenever $f \in C(\partial \Omega)$ and a bounded function g on $\partial \Omega$ such that g = f on $\partial \Omega \setminus E$, g is resolutive and $H_g = H_f$.

We can observe that if $E \subset \partial\Omega$ is a perturbation set of Ω , then every $x \in E$ is a perturbation point of Ω and $\omega_p(E,\Omega) = 0$ by letting f = 0 and $g = \chi_E$. Theorem 2.19 shows that if $E \subset \partial\Omega$ is of absolute p-harmonic measure zero(or equivalently of p-capacity zero), then E is a perturbation set. The following definition gives a notion which is similar to a set of absolute p-harmonic measure zero.

Definition 5.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that $E \subset \partial \Omega$ is of Ω -absolute p-harmonic measure zero if $\omega_p(E \cap \partial \tilde{\Omega}, \tilde{\Omega}) = 0$ for all bounded domains $\tilde{\Omega}$ such that $\tilde{\Omega} \cap U = \Omega \cap U$ for some open neighborhood U of E.

The following theorem shows a link between a perturbation set and a set of p-harmonic measure zero as well as characterizes a set of p-harmonic measure zero.

Theorem 5.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $E \subset \partial \Omega$ be a countable set. When 1 , the following conditions are equivalent.

- i) Every $x \in E$ is a perturbation point of Ω .
- ii) E is a perturbation set of Ω .
- iii) Whenever $\tilde{\Omega}$ is a bounded domain such that $\tilde{\Omega} \cap U = \Omega \cap U$ for some open neighborhood U of E, E is a perturbation set of $\tilde{\Omega}$.
- iv) For all $x \in E$, $\omega_n(\{x\}, \Omega) = 0$.
- v) $\omega_p(E,\Omega)=0$.
- vi) E is of Ω -absolute p-harmonic measure zero.

Furthermore, when p > n, the following conditions are also equivalent to i) $\sim vi$).

- vii) Every $x \in E$ is a game-perturbation point of Ω .
- viii) Every $x \in E$ is not an isolated boundary point.

Proof. Let $1 . To show the equivalence of <math>i) \sim vi$), note that it follows from the definitions that $iii) \Rightarrow ii) \Rightarrow i$ and $iii) \Rightarrow vi) \Rightarrow v) \Rightarrow iv$. Since Theorem 4.3 shows $iv) \Rightarrow i$), it is only need to show that $i) \Rightarrow iii$). Assume i). Theorem 4.5 shows that every $x \in E$ is also a perturbation point of $\tilde{\Omega}$, therefore Theorem 2.23 implies that E is a perturbation set of $\tilde{\Omega}$. When p > n, the equivalence of i) $\sim viii$) follows from Theorem 4.1.

Remark: Note that when $1 , <math>i) \sim vi$) are all true. Therefore, Theorem 5.3 is of special interest when p > n. $i) \sim iii$) is for a boundary perturbation problem and $iv) \sim vi$) is for p-harmonic measure. $i) \Leftrightarrow iv$) is the repetition of Theorem 4.3. $ii) \Leftrightarrow v$) is a generalization of Theorem 4.3. Both iii) and vi) show the locality of a boundary perturbation problem and p-harmonic measure when E is countable. Compare vi) to iii) in Proposition 2.13. When p > n, viii) provides a geometric criterion to show that E is a perturbation set of Ω or equivalently $\omega_p(E,\Omega) = 0$.

Theorem 5.4. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded domain. For each $k \in \mathbb{N}$, assume that $E_k \subset \partial \Omega$ is a countable set and $\omega_p(E_k, \Omega) = 0$. Then

$$\omega_p(\cup_k E_k, \Omega) = 0.$$

Proof. Let $x \in \bigcup_k E_k$. Since $\omega_p(\bigcup_k E_k, \Omega) = 0$, $\omega_p(\{x\}, \Omega) = 0$. Therefore the result follows from $iv \Leftrightarrow v$ in Theorem 5.3.

Next we give an invariance result for p-harmonic measure.

Theorem 5.5. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that $E \subset \partial \Omega$ is a countable set with $\omega_p(E,\Omega) = 0$. Then for every closed set $F \subset \partial \Omega$,

$$\omega_p(x; E \cup F, \Omega) = \omega_p(x; F, \Omega) \text{ for all } x \in \Omega.$$

Proof. It suffices to show that $\omega_p(x; F, \Omega) \geq \omega_p(x; E \cup F, \Omega)$. We can approximate χ_F by a decreasing sequence of continuous function $\{f_n\}$ such that $\lim_n f_n = \chi_F$ on $\partial\Omega$. Proposition 2.8 shows that $\lim_n H_{f_n}(x) = \omega_p(x; F, \Omega)$ for all $x \in \Omega$. Note that E is a perturbation set of Ω by Theorem 5.3, thereby $H_{f_n}(x) = H_{f_n+\chi_E}(x)$. Since $H_{f_n+\chi_E}(x) \geq \omega_p(x; E \cup F, \Omega)$, letting $n \to \infty$ shows that $\omega_p(x; F, \Omega) \geq \omega_p(x; E \cup F, \Omega)$. \square

Remark: When 1 , Kurki [Kur95] proved a similar invariance result by assuming that <math>E is a set of p-capacity zero instead of a countable set of p-harmonic measure zero. However, as the author is aware, Theorem 5.5 is a first invariance result for p-harmonic measure when p > n.

At last, we give a partial answer to Open problem 1.1 in some extreme cases; for any two closed subsets $E, F \subset \partial\Omega$ with $\omega_p(x; E, \Omega) = \omega_p(x; F, \Omega) = 0$, $\omega_p(x; E \cup F, \Omega) = 0$ if E and F are either somewhat "heavily" overlapped or "slightly" overlapped. The latter case is a slight generalization of Theorem 2.14.

Theorem 5.6. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded regular domain. Let $E, F \subset \partial \Omega$ are closed sets of p-harmonic measure zero. Further assume that either $(E \cup F) \setminus (E \cap F)$ is countable or there exists a closed set $G \subset \partial \Omega$ such that $G \subset F \setminus E$ and $F \setminus G$ is countable. Then $\omega_p(E \cup F, \Omega) = 0$.

Proof. Since $E \cup F = (E \cap F) \cup \{(E \cup F) \setminus (E \cap F)\}$ and $E \cap F$ is a closed set of p-harmonic measure zero, the result follows from Theorem 5.5. For ii) note that E and G are two disjoint closed sets of p-harmonic zero. Theorem 2.14 shows $\omega_p(E \cup G, \Omega) = 0$. Since $E \cup F = (E \cup G) \cup (F \setminus G)$ and $F \setminus G$ is a countable set of p-harmonic measure zero, the result follows again from Theorem 5.5.

6. Open problems

Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be a bounded domain throughout this section. It is easy to check that if $E \subset \partial \Omega$ is a perturbation set, E is of p-harmonic measure zero. When E is a countable set of p-harmonic measure zero, Theorem 5.3 shows that the converse is also true, i.e. if E is of p-harmonic measure zero, then E is a perturbation set. We may wonder whether this is still true when E is not a countable set.

Open Problem 6.1. If $E \subset \partial \Omega$ is of *p*-harmonic measure zero, then is E a perturbation set?

Let us recall that Theorem 2.18 and Theorem 2.19 show that if $E \subset \partial \Omega$ is of absolute p-harmonic measure zero or equivalently of p-capacity zero, then E is a perturbation set of Ω . The converse is generally not true. However, when E is a countable set, Theorem 5.3 shows that E is a perturbation set of Ω if and only if E is of Ω -absolute p-harmonic measure zero. This fact makes us conjecture the following question.

Open Problem 6.2. Is it true that $E \subset \partial \Omega$ is a perturbation set if and only if E is of Ω -absolute p-harmonic measure zero?

If the answers to the above two problems are yes, we can give an affirmative answer to the following open problem.

Open Problem 6.3. If $E \subset \partial \Omega$ is of *p*-harmonic measure zero, then is E of Ω -absolute *p*-harmonic measure zero?

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